# DETERMINING SIMPLICITY
## AND COMPUTING TOPOLOGICAL CHANGE
## IN STRONGLY NORMAL PARTIAL TILINGS
## OF $R^2$ OR $R^3$

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# DETERMINING SIMPLICITY
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## Abstract

A convex polygon in $R^2$, or a convex polyhedron in $R^3$, will be called a *tile*. A connected set $\mathcal{P}$ of tiles is called a *partial tiling* if the intersection of any two of the tiles is either empty, or is a vertex or edge (in $R^3$: or face) of both. $\mathcal{P}$ is called *strongly normal* (SN) if, for any partial tiling $\mathcal{P}' \subseteq \mathcal{P}$ and any tile $P \in \mathcal{P}'$, the neighborhood $N(P, \mathcal{P})$ of $P$ (the union of the tiles of $\mathcal{P}'$ that intersect $P$) is simply connected. Let $\mathcal{P}$ be SN, and let $N^*(P, \mathcal{P})$ be the excluded neighborhood of $P$ in $\mathcal{P}$ (i.e., the union of the tiles of $\mathcal{P}$, other than $P$ itself, that intersect $P$). We call $P$ *simple* in $\mathcal{P}$ if $N(P, \mathcal{P})$ and $N^*(P, \mathcal{P})$ are topologically equivalent. This paper presents methods of determining, for an SN partial tiling $P$, whether a tile $P \in \mathcal{P}'$ is simple, and if not, of counting the numbers of components and holes (in $R^3$: components, tunnels and cavities) in $N^*(P, \mathcal{P})$.

# 1 Introduction

In $n$-dimensional Euclidean space $R^n$, a *convex hyperpolyhedron* $P$ is a bounded set which is the intersection of a finite number of closed half-spaces and which has a nonempty interior. Let $H_1, \cdots, H_m$ be the hyperplanes (($n-1$)-dimensional subspaces of $R^n$) that bound the half-spaces whose intersection is $P$. It can be shown that $m$ must be at least $n+1$. Each nonempty $P \cap H_i$ is called an ($n-1$)-dimensional *hyperface* of $P$. An ($n-1$)-dimensional hyperface of $P$ is a convex hyperpolyhedron in $R^{n-1}$; its ($n-2$)-dimensional hyperfaces are also called ($n-2$)-dimensional hyperfaces of $P$, and so on. For $n - d = 2, 1$, or $0$, an ($n-d$)-dimensional hyperface of $P$ is called a *face*, an *edge*, or a *vertex* of $P$, respectively. It can be shown that a face is a convex planar polygon, an edge is a line segment, and a vertex is a point.

Let $\mathcal{P}$ be a nonempty set of convex hyperpolyhedra whose union is connected. We call $\mathcal{P}$ *normal* if, for all $P, Q \in \mathcal{P}$, $P \cap Q$ is either empty or an ($n-d$)-dimensional hyperface of $P$ and of $Q$, for some $d$. [For example, when $n = 2$, the hyperpolyhedra are polygons; if a set of convex polygons is normal, the intersection of any two of them is an edge, a vertex, or empty; their intersection cannot have an interior point, and if it is contained in an edge, it must either be the entire edge or one of the vertices of the edge.] If $\mathcal{P}$ is normal and its union is all of $R^n$, it is called a *tessellation* or *tiling* of $R^n$; if its union is not all of $R^n$ it is called a *partial tiling* of $R^n$. In either case, the $P$'s in $\mathcal{P}$ are called *tiles*. It is not hard to see that any partial tiling is a subset of tiling, because the space not occupied by the tiles can be partitioned into convex polyhedra whose pairwise intersections are hyperfaces.

The *neighborhood* $N(P, \mathcal{P})$ of $P$ in $\mathcal{P}$ is the union of all $Q \in \mathcal{P}$ that intersect $P$ (including $P$ itself). The *excluded neighborhood* $N^*(P, \mathcal{P})$ is defined similarly, except that $Q$ is not allowed to be $P$ itself.

$\mathcal{P}$ is called *strongly normal* (SN) if it is normal and, for all $P, P_1, \ldots, P_m$ $(m > 1) \in \mathcal{P}$, if each $P_i$ intersects $P$ and $I = P_1 \cap \ldots \cap P_m$ is nonempty, then $I$ intersects $P$. Note that both normality and strong normality are hereditary: If they hold for $\mathcal{P}$, they hold for any $\mathcal{P}' \subseteq \mathcal{P}$. It was observed in [1] that the regular square and hexagonal tessellations of $R^2$ are SN, but the regular triangular tessellation is not.

¿From now on we assume that $n = 2$ or $3$, so that the $P$'s are convex polygons or polyhedra. In [1] we showed that $\mathcal{P}$ is an SN (partial) tiling of $R^2$ or $R^3$ by convex polygons or polyhedra iff, for any $\mathcal{P}' \subseteq \mathcal{P}$ and any $P \in \mathcal{P}'$, $N(P, \mathcal{P}')$ is simply connected.

If $\mathcal{P}$ is SN and $N^*(P, \mathcal{P})$ is simply connected, $P$ is called *simple* in $\mathcal{P}$. When $P$ is not simple, $N^*(P, \mathcal{P})$ may be disconnected, or it may have holes (in 2D), tunnels or cavities (in 3D). In this paper we present methods of determining whether $P$ is simple, and if not, of counting the numbers of connected components, holes, tunnels, and cavities (as appropriate) in $N^*(P, \mathcal{P})$. From now on we assume that $\mathcal{P}$ is SN. To avoid the need for awkward formulations or artificial terminology, we treat the two- and three-dimensional cases separately (in Sections 2 and 3, respectively), even though their treatments are to a great extent analogous.

## 2 The two-dimensional case

### 2.1 Numbers of components and holes

The interior of a polygon $P$, denoted by $interior(P)$, is defined as $P -$ (the union of the edges of $P$). If $e$ is an edge, we define $interior(e)$ as $e -$ (the vertices of $e$). It will be convenient, in what follows, to define $interior(v)$ as $v$ if $v$ is a vertex.

A vertex or edge $s$ of $\mathcal{P}$ is called *shared* if there exist two polygons $P_i, P_j \in \mathcal{P}$ such that $P_i \cap P_j = s$. Otherwise, $s$ is called *bare*. An edge that belongs two polygons must be shared, but a bare vertex $v$ can belong to two polygons if they intersect in an edge containing $v$.

The *boundary* of $P$, denoted by $\mathcal{B}(P)$, is the union of all edges of $P$. The set $\mathcal{B}(P) - N^*(P, \mathcal{P})$, denoted by $\mathcal{B}_b(P, \mathcal{P})$, is called the *bare boundary* of $P$. The union of the shared vertices and edges of $P$ is called the *shared boundary* of $P$ and is denoted by $\mathcal{B}_s(P, \mathcal{P})$. It is not hard to see that for all $P$, $\mathcal{B}_b(P, \mathcal{P})$ and $\mathcal{B}_s(P, \mathcal{P})$ are disjoint and their union is $\mathcal{B}(P)$. Note that $\mathcal{B}_b(P, \mathcal{P})$ is not the union of the bare vertices and edges of $P$; a vertex of a shared edge may be bare, and a vertex of a bare edge may be shared. An edge of $P$ is shared iff one of the polygons sharing it is $P$; but a vertex of $P$ may be shared by two polygons neither of which is $P$.

If an edge of $P$ is bare, its interior (at least) is contained in $\mathcal{B}_b(P, \mathcal{P})$; and it is not hard to see that if every edge of $P$ is shared, $\mathcal{B}_b(P, \mathcal{P})$ must be empty. This proves

**Lemma 1** $\mathcal{B}_b(P, \mathcal{P})$ *is non-empty if and only if $P$ has a bare edge.*

The union of all the polygons in $\mathcal{P}$ will be denoted by $\mathcal{U}(\mathcal{P})$. The numbers of components and holes in $\mathcal{U}(\mathcal{P})$ are its 0th and 1st Betti numbers, respectively; its higher Betti numbers are zero. As shown in [1], because of SN, $N(P, \mathcal{P})$ always consists of a single connected component and has no holes. In this section we will determine the numbers of components and holes in $N^*(P, \mathcal{P})$. We first establish a useful lemma:

**Lemma 2** *The numbers of components and holes in $N^*(P, \mathcal{P})$ are, respectively, the same as the numbers of components and holes in $\mathcal{B}_s(P, \mathcal{P})$.*

**Proof:** Let $disc(\Delta, p)$ denote the disc with radius $\Delta$ and center $p$ ($p \in R^2$). For every point $p \in N(P, \mathcal{P}) - P$ there exists a finite value $d$ such that for all $\Delta < d$, $disc(\Delta, p) \cap N(P, \mathcal{P}) = disc(\Delta, p) \cap N^*(P, \mathcal{P})$. Now $P$ and $N(P, \mathcal{P})$ are both simply connected and $P \subset N(P, \mathcal{P})$. Therefore, there exists a Betti number preserving transformation (a deformation retract) from $N(P, \mathcal{P})$ to $P$. This transformation basically removes $N(P, \mathcal{P}) - P$ from $N(P, \mathcal{P})$. The transformation is also Betti number preserving for $N^*(P, \mathcal{P})$. This is true because: (a) The transformation removes $N(P, \mathcal{P}) - P$; (b) for every point $p \in N(P, \mathcal{P}) - P$ there exists a finite value $d$ such that for all $\Delta < d$, $disc(\Delta, p) \cap N(P, \mathcal{P}) = disc(\Delta, p) \cap N^*(P, \mathcal{P})$; (c) both $N(P, \mathcal{P})$ and $P$ are simply connected and thus the transformation is Betti number preserving throughout (i.e. it is not the case that at a certain step it removes a hole and at some other step it creates a hole). Since the transformation removes $N(P, \mathcal{P}) - P$, it transforms $N^*(P, \mathcal{P})$ to $N^*(P, \mathcal{P}) - (N(P, \mathcal{P}) - P) = N^*(P, \mathcal{P}) \cap P = N^*(P, \mathcal{P}) \cap \mathcal{B}(P) = \mathcal{B}_s(P, \mathcal{P})$.

Therefore the numbers of components and holes in $N^*(P, \mathcal{P})$ are the same as those numbers in $\mathcal{B}_s(P, \mathcal{P})$. $\qquad\square$

Now $\mathcal{B}_s(P, \mathcal{P})$ is a subset of $\mathcal{B}(P)$ and $\mathcal{B}(P)$ is topologically equivalent to a circle, which has exactly one hole. Therefore $\mathcal{B}_s(P, \mathcal{P})$ can have at most one hole, and this occurs only when $\mathcal{B}_s(P, \mathcal{P})$ is topologically equivalent to a circle. In other words, $\mathcal{B}_s(P, \mathcal{P})$ has at most one hole, and this occurs only when $\mathcal{B}_s(P, \mathcal{P}) = \mathcal{B}(P)$, i.e. $\mathcal{B}_b(P, \mathcal{P})$ is empty. Based on this discussion we have

**Proposition 1** $N^*(P, \mathcal{P})$ *can have at most one hole, and this occurs only when $\mathcal{B}_b(P, \mathcal{P})$ is empty.*

¿From Proposition 1 and Lemma 1 we have

**Proposition 2** $N^*(P, \mathcal{P})$ *can have at most one hole, and this occurs if and only if all edges of $P$ are shared.*

## 2.2   Simple polygons and measures of topological change

A polygon $P$ is *simple* in $\mathcal{P}$ if and only if deleting $P$ from $\mathcal{P}$ does not change the topology of $\mathcal{U}(\mathcal{P})$. Evidently, deleting $P$ changes the topology of $\mathcal{U}(\mathcal{P})$ if and only if it changes the topology of its neighborhood (i.e. the union of the polygons that intersect it). Thus $P$ is simple if and only if the numbers of components and holes are the same in $N(P, \mathcal{P})$ and in $N^*(P, \mathcal{P})$.

As shown in [1], since $\mathcal{P}$ is SN, the numbers of components and holes in $N(P, \mathcal{P})$ are always 1 and 0 respectively. Therefore $P$ is simple if and only if $N^*(P, \mathcal{P})$ has exactly one component and has no holes.

Criteria for the existence of holes in $N^*(P, \mathcal{P})$ were given in Section 2.1. These criteria give us the following characterization of a simple polygon:

**Theorem 1** *$P$ is simple if and only if it satisfies the following two conditions:*

*1. $B_s(P, \mathcal{P})$ has exactly one component.*

*2. $P$ has a bare edge.*

In general, we can define $P$ to be simple if $N(P, \mathcal{P})$ can be continuously contracted into $N^*(P, \mathcal{P})$, or equivalently, if $P$ can be continuously contracted into $B_s(P, \mathcal{P})$. It can be shown that $P$ is simple iff $N^*(P, \mathcal{P})$ has the same numbers of components and holes as $N(P, \mathcal{P})$, and that Theorem 1 holds even if $\mathcal{P}$ is normal (but not necessarily SN).

If a polygon is non-simple, its deletion changes the topology of $\mathcal{U}(\mathcal{P})$, but this tells us nothing about the nature of the change. The following theorem is a straightforward consequence of the discussion in Section 2.1.

**Theorem 2** *When $P$ is deleted,*

3

*1. The change in the number of components is one less than the number of components of $\mathcal{B}_s(P, \mathcal{P})$.*

*2. The change in the number of holes is one when $P$ has no bare edge and zero otherwise.*

Let the *configuration* of an edge or vertex refer to whether it is shared or bare. By Theorems 1 and 2, the changes in the numbers of components and holes when $P$ is deleted (and hence the simplicity of $P$) are determined if we know the configurations of the vertices and edges of $P$. [Note that this is not true if $\mathcal{P}$ is not SN (even if it is normal). For example, the configurations of the vertices and edges of the triangle $P$ in Figures 1a and 1b are the same, but in Figure 1a $N^*(P, \mathcal{P})$ has only one component, while in Figure 1b it has two components.]

An alternative formulation of Theorems 1 and 2 can be obtained by regarding the given partial tiling $\mathcal{P}'$ as a subset of a tiling $\mathcal{P}$, and distinguishing the polygons $Q$ in $N^*(P, \mathcal{P})$ that share an edge with $P$ from those that only share a vertex with $P$. The former polygons will be called $\beta$-adjacent to $P$, while all the polygons in $N^*(P, \mathcal{P})$ will be called $\alpha$-adjacent to $P$. [As usual, the transitive closure of $x$-adjacency (where $x = \alpha$ or $\beta$) is called $x$-connectedness.] We denote the set of polygons in $\mathcal{P}'$ that are $\alpha$-adjacent to $P$ by $N_\alpha^*(P)$, and the set of polygons in $\mathcal{P} - \mathcal{P}'$ that are $\beta$-adjacent to $P$ by $\overline{N}_\beta^*(P)$. In terms of this notation, we can restate Theorems 1 and 2 as follows:

**Theorem 1'**  *P is simple if and only if it satisfies the following two conditions:*

*1. $N_\alpha^*(P)$ is nonempty and $\alpha$-connected.*

*2. $\overline{N}_\beta^*(P)$ is nonempty.*

**Theorem 2'**  *When $P$ is deleted,*

*1. The change in the number of components is one less than the number of $\alpha$-components in $N_\alpha^*(P)$.*

*2. The change in the number of holes is one when $\overline{N}_\beta^*(P)$ is nonempty, and zero otherwise.*

### 2.3  Efficient computation

As we have seen, the numbers of components and holes in $N(P, \mathcal{P})$ are always 1 and 0 respectively. Thus the numbers $\xi(P, \mathcal{P})$ and $\delta(P, \mathcal{P})$ of components and holes in $N^*(P, \mathcal{P})$ define local measures of topological change when $P$ is deleted. Let $simple(P, \mathcal{P})$ be a predicate which has the value 1 when $P$ is simple in $\mathcal{P}$ and 0 otherwise. $\xi(P, \mathcal{P})$, $\delta(P, \mathcal{P})$ and $simple(P, \mathcal{P})$ will be referred to as the local topological parameters of $P$. In this section we develop an efficient approach to computing these parameters.

**Proposition 3** *If an edge $e$ of $P$ is shared then $\xi(P, \mathcal{P})$ is independent of the configurations of the vertices $v \subset e$.*

**Proof:** Let $\mathcal{P}$ contain a polygon $P_v$ such that $P_v \cap P = v$. To establish the proposition we shall show that the numbers of components of $N^*(P, \mathcal{P})$ and $N^*(P, \mathcal{P} - \{P_v\})$ are the same. If this is not true then one of the following two cases must occur:

**Case 1:** $P_v$ is an isolated polygon in $N^*(P, \mathcal{P})$, so that the number of components of $N^*(P, \mathcal{P})$ is greater than that of $N^*(P, \mathcal{P} - \{P_v\})$.

**Case 2:** Two or more components of $N^*(P, \mathcal{P} - \{P_v\})$ are adjacent to $P_v$, so that the number of components of $N^*(P, \mathcal{P} - \{P_v\})$ is greater than that of $N^*(P, \mathcal{P})$.

Since $e$ is shared, there exists a polygon $P_e \subset N^*(P, \mathcal{P})$ such that $P_e \cap P = e$. Now $P_v \cap P = v$ implies that $v \subset P_v$. Also $v \subset e \subset P_e$ and thus $v \subset P_e \cap P_v$, i.e. $P_e \cap P_v \neq \emptyset$. Hence $P_v$ is adjacent to $P_e$ and Case 1 never occurs. As regards Case 2, suppose there exist two polygons $P_1, P_2 \subset N^*(P, \mathcal{P} - \{P_v\})$ that belong to different components of $N^*(P, \mathcal{P} - \{P_v\})$ and are adjacent to $P_v$. $P_1$ and $P_v$ are adjacent to $P$ and $P_1 \cap P_v \neq \emptyset$; thus by SN, $(P_1 \cap P_v) \cap P \neq \emptyset$. Obviously, $(P_1 \cap P_v) \cap P \subset P_v \cap P = v$ and thus $P_1 \cap P_v \cap P \subset v \subset e \subset P_e$, which implies that $P_1 \cap P_e \neq \emptyset$; hence $P_1$ and $P_e$ are adjacent. In the same way, it can be shown that $P_2$ and $P_e$ are adjacent. Therefore $P_1 \cup P_e \cup P_2$ is connected, i.e. $P_1$ and $P_2$ belong to the same component of $N^*(P, \mathcal{P} - \{P_v\})$, contradiction. Thus Case 2 cannot occur either. $\square$

By Proposition 2, $\delta(P, \mathcal{P})$ depends on the configurations of the edges of $P$, and is independent of the configurations of the vertices of $P$. By Proposition 3, $\xi(P, \mathcal{P})$ is independent of the configurations of the vertices belonging to shared edges of $P$. Based on this, we have the following definition and Proposition:

**Definition 1** *A vertex $v$ of $P$ is called* trapped *if it belongs to a shared edge of $P$; otherwise, it is called* free.

**Proposition 4** $\xi(P, \mathcal{P})$ *and* $\delta(P, \mathcal{P})$ *are independent of the configurations of the trapped vertices of $P$.*

Since $simple(P, \mathcal{P})$ depends on $\xi(P, \mathcal{P})$ and $\delta(P, \mathcal{P})$ (specifically, $simple(P, \mathcal{P})$ is 1 if and only if $\xi(P, \mathcal{P})$, $\delta(P, \mathcal{P})$ are 1 and 0, respectively), we also have

**Corollary 1** *The predicate* simple $(P, \mathcal{P})$ *is independent of the configurations of the trapped vertices of $P$.*

**Lemma 3** *Each free, shared vertex of $P$ contributes one component to $N^*(P, \mathcal{P})$.*

**Proof:** Let $v$ be a free, shared vertex of $P$. Let $P_1, P_2, \cdots, P_n$ be the polygons of $\mathcal{P}$ such that $P_i \cap P = v$. [Since $v$ is shared, it is in at least one polygon $P'$ other than $P$. Now $P' \cap P$ cannot be an edge, since $v$ is free; hence $P' \cap P = v$.] Evidently, the $P_i$'s are connected. We show that no other polygon in $N(P, \mathcal{P})$ meets any of the $P_i$'s. In fact, suppose $P'$ is not one of the $P_i$'s (so that $P' \cap P \neq v$), but $P'$ meets some $P_i$, i.e. $P' \cap P_i \neq \emptyset$. Since $v$ is free, both of the edges of $P$ that meet at $v$ are bare; hence $(P' \cap P_i) \cap P = \emptyset$. But this contradicts the fact that $\mathcal{P}$ is SN. $\square$

Using these results, we now describe a simple method of computing the local topological parameters $\xi(P, \mathcal{P})$, $\delta(P, \mathcal{P})$, and $simple(P, \mathcal{P})$. We treat three cases, depending on the presence of shared edges in $P$.

5

**Case 1:** All edges of $P$ are shared.

In this case all vertices are trapped and no more computation is needed. The local topological parameters have the following values:

$$\xi(P, \mathcal{P}) = 1, \quad \delta(P, \mathcal{P}) = 1, \quad simple(P, \mathcal{P}) = 0.$$

**Case 2:** Some, but not all, edges of $P$ are shared.

In this case $\delta(P, \mathcal{P})$ is always zero, but $\xi(P, \mathcal{P})$ (and therefore $simple(P, \mathcal{P})$) depends on the configurations of the edges and free vertices (if any) of $P$. We can compute $\xi(P, \mathcal{P})$ by examining the successive edges and counting "1" for each pair of consecutive edges one of which is bare and the other shared, and counting "1" for each shared vertex which is the intersection of two bare edges. Finally, $simple(P, \mathcal{P})$ is 1 if $\xi(P, \mathcal{P})$ is 1, and is 0 otherwise.

**Case 3:** No edge of $P$ is shared.

In this case all vertices of $P$ are free. The number of holes $\delta(P, \mathcal{P})$ is always zero; the number of components $\xi(P, \mathcal{P})$ is exactly the same as the number of shared vertices; and $simple(P, \mathcal{P})$ is 1 when exactly one vertex is shared, and is 0 otherwise.

## 3 The three-dimensional case

### 3.1 Numbers of components, tunnels and cavities

The interior of a polyhedron $P$, denoted by $interior(P)$, is defined as $P - $ (the union of the faces of $P$). If $f$ is a face, we define $interior(f)$ as $f - $ (the edges of $f$). The interior of an edge or a vertex is defined as in the two-dimensional case.

A vertex, edge, or face $s$ of $\mathcal{P}$ is called *shared* if there exist two polyhedra $P_i, P_j \in \mathcal{P}$ such that $P_i \cap P_j = s$. Otherwise, $s$ is called *bare*. A face that belongs to two polyhedra must be shared, but a bare edge (respectively, vertex) $x$ can belong to two polyhedra if they intersect in a face (respectively, face or edge) containing $x$.

The *boundary* of $P$, denoted by $\mathcal{B}(P)$, is the union of all faces of $P$. The set $\mathcal{B}(P) - N^*(P, \mathcal{P})$, denoted by $\mathcal{B}_b(P, \mathcal{P})$, is called the *bare boundary* of $P$. The union of the shared vertices, edges, and faces of $P$ is called the *shared boundary* of $P$ and is denoted by $\mathcal{B}_s(P, \mathcal{P})$. It is not hard to see that for all $P$, $\mathcal{B}_b(P, \mathcal{P})$ and $\mathcal{B}_s(P, \mathcal{P})$ are disjoint and their union is $\mathcal{B}(P)$. Note that $\mathcal{B}_b(P, \mathcal{P})$ is not the union of the bare vertices, edges, and faces of $P$; a vertex of a shared face or shared edge may be bare; an edge of a shared face may be bare; a vertex of a bare edge or bare face may be shared; and an edge of a bare face may be shared. A face of $P$ is shared iff one of the polyhedra sharing it is $P$; but a vertex or edge of $P$ may be shared by two polyhedra neither of which is $P$.

If a face of $P$ is bare, its interior (at least) is contained in $\mathcal{B}_b(P, \mathcal{P})$; and it is not hard to see that if every face of $P$ is shared, $\mathcal{B}_b(P, \mathcal{P})$ must be empty. This proves

**Lemma 4** $\mathcal{B}_b(P, \mathcal{P})$ *is non-empty if and only if $P$ has a bare face.*

The union of all the polyhedra in $\mathcal{P}$ will be denoted by $\mathcal{U}(\mathcal{P})$. The numbers of components, tunnels and cavities in $\mathcal{U}(\mathcal{P})$ are its 0th, 1st and 2nd Betti numbers, respectively. As shown

in [1], because of SN, $N(P,\mathcal{P})$ always consists of a single connected component without tunnels and cavities. In this section we will determine the numbers of components, tunnels and cavities in $N^*(P,\mathcal{P})$. We first establish a useful lemma:

**Lemma 5** *The numbers of components, tunnels and cavities in $N^*(P,\mathcal{P})$ are the same as the numbers of components, tunnels and cavities in $\mathcal{B}_s(P,\mathcal{P})$, respectively.*

**Proof:** Analogous to that of Lemma 2, using a ball instead of a disc. □

$\mathcal{B}_s(P,\mathcal{P})$ is a subset of $\mathcal{B}(P)$, and $\mathcal{B}(P)$ is topologically equivalent to a sphere, which has exactly one cavity. Therefore $\mathcal{B}_s(P,\mathcal{P})$ can have at most one cavity, and this occurs only when $\mathcal{B}_s(P,\mathcal{P})$ is topologically equivalent to a sphere. In other words, $\mathcal{B}_s(P,\mathcal{P})$ has at most one cavity, and this occurs only when $\mathcal{B}_s(P,\mathcal{P}) = \mathcal{B}(P)$, i.e. $\mathcal{B}_b(P,\mathcal{P})$ is empty. Based on this discussion we have

**Proposition 5** *$N^*(P,\mathcal{P})$ can have at most one cavity, and this occurs only when $\mathcal{B}_b(P,\mathcal{P})$ is empty.*

¿From Proposition 5 and Lemma 4 we have

**Proposition 6** *$N^*(P,\mathcal{P})$ can have at most one cavity, and this occurs if and only if all faces of $P$ are shared.*

The remainder of this section deals with methods of determining the number of tunnels in $N^*(P,\mathcal{P})$. Let $W$ be a connected subset of the boundary $B$ of a convex set in $R^3$ — for example, a connected region on the surface of a convex polyhedron. $W$ has a tunnel if and only if $B - W$ is not connected. In fact, $W$ has $n > 0$ tunnels if and only if $B - W$ has $n + 1$ components. Now $P$ is a convex set in $R^3$, $\mathcal{B}(P)$ is the boundary of $P$, and $\mathcal{B}_s(P,\mathcal{P})$ is a subset of $\mathcal{B}(P)$. Also $\mathcal{B}_b(P,\mathcal{P}) = \mathcal{B}(P) - N^*(P,\mathcal{P}) = \mathcal{B}(P) - N^*(P,\mathcal{P}) \cap \mathcal{B}(P) = \mathcal{B}(P) - \mathcal{B}_s(P,\mathcal{P})$. Thus $\mathcal{B}_s(P,\mathcal{P})$ has no tunnel when $\mathcal{B}_b(P,\mathcal{P})$ is connected or empty; otherwise, the number of tunnels in $\mathcal{B}_s(P,\mathcal{P})$ is one less than the number of components in $\mathcal{B}_b(P,\mathcal{P})$. According to Lemma 5, the number of tunnels in $N^*(P,\mathcal{P})$ is equal to the number in $\mathcal{B}_s(P,\mathcal{P})$. Based on this discussion we have

**Proposition 7** *$N^*(P,\mathcal{P})$ has no tunnel when $\mathcal{B}_b(P,\mathcal{P})$ is connected or empty; otherwise, the number of tunnels in $N^*(P,\mathcal{P})$ is one less than the number of components in $\mathcal{B}_b(P,\mathcal{P})$.*

If $f$ is a bare face of $P$, $interior(f)$ must be contained in $\mathcal{B}_b(P,\mathcal{P})$, and since $interior(f)$ is connected, it must be a subset of some component of $\mathcal{B}_b(P,\mathcal{P})$. Conversely, we have

**Lemma 6** *Every component of $\mathcal{B}_b(P,\mathcal{P})$ contains $interior(f)$ for some bare face $f$ of $P$.*

**Proof:** Suppose component $c$ of $\mathcal{B}_b(P,\mathcal{P})$ does not contain the interior of any bare face of $P$. Then $c$ must contain the interior of some bare edge or bare vertex. Suppose $c$ contains $interior(e)$ for some bare edge $e$. Let $e$ be a subset of the face $f$. If $f$ is bare, $interior(f)$ is a subset of $\mathcal{B}_b(P,\mathcal{P})$. But $interior(f) \cup interior(e)$ is connected, so must be contained in $c$, contradiction. On the other hand, if $f$ is not bare, it is a subset of $N^*(P,\mathcal{P})$. Hence $interior(e) \subset e \subset f \subset N^*(P,\mathcal{P})$ cannot be a subset of $\mathcal{B}_b(P,\mathcal{P})$, contradiction. The proof if $c$ contains the interior of some bare vertex is analogous. □

7

**Lemma 7** *For any two bare faces $f, f'$ of $P$, interior$(f)$ and interior$(f')$ are connected in $\mathcal{B}_b(P, \mathcal{P})$ if and only if there exists an alternating sequence $f = f_0, e_0, f_1, \cdots, f_i, e_i, f_{i+1}, \cdots, e_{n-1}, f_n = f'$ of bare faces and bare edges of $P$ such that $e_i = f_i \cap f_{i+1}$ for $0 \leq i < n$.*

**Proof:** Suppose first that there exists such an alternating sequence. Since $e_i$ is a subset of both $f_i$ and $f_{i+1}$, interior$(f_i) \cup$ interior$(e_i) \cup$ interior$(f_{i+1})$ is connected, and obviously interior$(f_i) \cup$ interior$(e_i) \cup$ interior$(f_{i+1})$ is a subset of $\mathcal{B}_b(P, \mathcal{P})$. Thus interior$(f_i)$ and interior$(f_{i+1})$ are connected in $\mathcal{B}_b(P, \mathcal{P})$ for $0 \leq i < n$, and thus interior$(f) =$ interior$(f_0)$ and interior$(f') =$ interior$(f_n)$ are connected in $\mathcal{B}_b(P, \mathcal{P})$. Hence if there exists such an alternating sequence, interior$(f)$ and interior$(f')$ are connected in $\mathcal{B}_b(P, \mathcal{P})$.

We shall show that, conversely, if interior$(f)$ and interior$(f')$ are connected in $\mathcal{B}_b(P, \mathcal{P})$, there exists such an alternating sequence. A curve connecting them in $\mathcal{B}_b(P, \mathcal{P})$ passes through the interiors of a sequence $f = s_0, s_1, \cdots, s_m = f'$ of bare simplexes of $P$. Since the interiors of the $s_i$'s intersect $\mathcal{B}_b(P, \mathcal{P})$, by normality, it is not hard to see that they must be subsets of $\mathcal{B}_b(P, \mathcal{P})$; hence the $s_i$'s are bare simplexes. Evidently interior$(s_i) \cup$ interior$(s_{i+1})$ must be connected for $0 \leq i < n$, since the curve passes through the consecutive $s_i$'s. Now the unions of the interiors of two faces, edges, or vertices of a polyhedron can never be connected. Thus, if we have a sequence $x_0, x_1, \cdots, x_k$ of faces and edges of a polyhedron in which interior$(x_i) \cup$ interior$(x_{i+1})$ is connected for $0 \leq i < k$, it must be an alternating sequence of faces and edges. Hence it remains only to show that we can replace every vertex in the sequence $s_0, s_1, \cdots, s_m$ by a sequence of bare faces and bare edges which satisfy the connectivity condition.

A vertex in $s_0, s_1, \cdots, s_m$ must occur in one of the following four contexts (note that $s_0$ and $s_m$ are not vertices):

(a) face, vertex, edge
(b) edge, vertex, face
(c) face, vertex, face
(d) edge, vertex, edge.

We show how to do the replacement in case (a); the other three cases can be treated similarly. Let $x, y, z$ be the face, the vertex and the edge, respectively. If $z \subset x$, i.e. the edge is a subset of the face, then interior$(x) \cup$ interior$(z)$ is connected and we can replace $x, y, z$ by $x, z$, which is an alternating sequence of bare faces and bare edges satisfying the connectivity condition. If $z \not\subset x$ then we proceed as follows: Since interior$(x) \cup$ interior$(y) \cup$ interior$(z)$ is connected and $y$ is a vertex, $y$ must be a subset of both $x$ and $z$. Since $y$ is a vertex of $P$, at least three faces and at least three edges of $P$ must meet at $v$; moreover, these $n \geq 3$ faces and edges can be arranged in a sequence such that the $i$th edge is the intersection of the $i$th face and the $(i+1)$st face $(\bmod n)$. Let the faces be $x_1, x_2, \cdots, x_n$ and the edges be $z_1, z_2, \cdots, z_n$, where $z_i = x_i \cap x_{i+1 \bmod n}$. Since $x$ and $z$ both meet $y$, $x$ is one of the $x_i$'s and $z$ is one of the $z_i$'s. Suppose $x = x_j$ and $z = z_k$ where $j > k$; we have already defined the replacement method for $j = k$, in which $z$ is a subset of $x$, and the replacement method for $j < k$ is similar to that for $j > k$. Since interior$(y)$ is a subset of $\mathcal{B}_b(P, \mathcal{P})$, $x_1, x_2, \cdots, x_n$ and $z_1, z_2, \cdots, z_n$ must all be bare simplexes. [Suppose this were not true, e.g. $x_1$ is not bare, so that it is a subset of $N^*(P, \mathcal{P})$. Then interior$(y) = y \subset x_1 \subset N^*(P, \mathcal{P})$; thus interior$(y) \not\subset \mathcal{B}(P) - N^*(P, \mathcal{P})$, i.e. interior$(y) \not\subset \mathcal{B}_b(P, \mathcal{P})$, contradiction.]

Now $z_i = x_i \cap x_{i+1 \bmod n}$ implies that $interior(x_i) \cup interior(z_i) \cup interior(x_{i+1 \bmod n})$ is connected; hence $interior(x_j) \cup interior(z_j) \cup interior(x_{j+1}) \cdots \cup interior(x_k) \cup interior(z_k)$ is connected. Therefore we can replace the sequence $x, y, z$ by the alternating sequence $x = x_j, z_j, x_{j+1}, \cdots, x_k, z_k$ of bare faces and bare edges which satisfies the connectivity condition. $\square$

**Definition 2** *Let $O(P, \mathcal{P})$ be the set of bare faces of $P$. Two bare faces $f, f'$ of $P$ are called* face-adjacent *if the edge $f \cap f'$ is bare. Two bare faces $f, f'$ of $P$ are called* face-connected *if there exists a sequence $f = f_0, f_1, \cdots, f_n = f'$ in $O(P, \mathcal{P})$ such that $f_i$ and $f_{i+1}$ are face-adjacent for $0 \leq i < n$. A* face component *of $O(P, \mathcal{P})$ is the union of a maximal face-connected subset of $O(P, \mathcal{P})$.*

**Theorem 3** *The number of tunnels in $N^*(P, \mathcal{P})$ is zero when $P$ has no bare face. Otherwise, the number of tunnels in $N^*(P, \mathcal{P})$ is one less than the number of face components in $O(P, \mathcal{P})$.*

**Proof:** This follows from Proposition 7 and Lemmas 6 and 7. $\square$

The following corollaries are straightforward consequences of this theorem:

**Corollary 2** *$N^*(P, \mathcal{P})$ contains no tunnel if and only if $O(P, \mathcal{P})$ is face-connected.*

**Corollary 3** *The number of tunnels in $N^*(P, \mathcal{P})$ is independent of whether the vertices of $P$ are bare or shared.*

## 3.2 Simple polyhedra and measures of topological change

A polyhedron $P$ is *simple* in $\mathcal{P}$ if and only if deleting $P$ from $\mathcal{P}$ does not change the topology of $\mathcal{U}(\mathcal{P})$. Evidently, deleting $P$ changes the topology of $\mathcal{U}(\mathcal{P})$ if and only if it changes the topology of its neighborhood (i.e. the union of the polyhedra that intersect it). Thus $P$ is simple if and only if the numbers of components, tunnels and cavities are the same in $N(P, \mathcal{P})$ and in $N^*(P, \mathcal{P})$.

As shown in [1], since $\mathcal{P}$ is SN, the numbers of components, tunnels and cavities in $N(P, \mathcal{P})$ are always 1, 0, and 0 respectively. Therefore $P$ is simple if and only if $N^*(P, \mathcal{P})$ has exactly one component and has no tunnels or cavities. Criteria for the existence of tunnels and cavities in $N^*(P, \mathcal{P})$ were given in Section 3.1 (Corollary 2 and Proposition 6). These criteria gives us the following characterization of a simple polyhedron.

**Theorem 4** *$P$ is simple if and only if it satisfies the following two conditions:*

1. *$\mathcal{B}_s(P, \mathcal{P})$ has exactly one component.*

2. *$O(P, \mathcal{P})$ is non-empty and face-connected.*

If we define "simple" in terms of contractability (see Section 2.2, just after Theorem 1), it can be shown that even if $\mathcal{P}$ is normal (but not necessarily SN), $P$ is simple iff conditions 1 and 2 of Theorem 4 hold. For such a $P$, $N^*(P, \mathcal{P})$ must have the same numbers of components, tunnels, and cavities as $N(P, \mathcal{P})$; but these numbers can also be the same even if $P$ is not simple (unlike the situation in two dimensions, where equality of the numbers of components and holes implies simplicity).

If a polyhedron is non-simple, its deletion changes the topology of $\mathcal{U}(\mathcal{P})$, but this tells us nothing about the nature of the change. The following theorem is a straightforward consequence of the discussion in Section 3.1.

**Theorem 5** *When $P$ is deleted,*

1. *The change in the number of components is one less than the number of components of $\mathcal{B}_s(P, \mathcal{P})$.*

2. *The change in the number of tunnels is one less than the number of face components in $O(P, \mathcal{P})$ when $O(P, \mathcal{P})$ is non-empty, and zero otherwise.*

3. *The change in the number of cavities is one when $O(P, \mathcal{P})$ is empty and zero otherwise.*

Let the *configuration* of a face, edge, or vertex refer to whether it is shared or bare. By Theorems 4 and 5, the changes in the numbers of components, tunnels, and cavities when $P$ is deleted (and hence the simplicity of $P$) are determined if we know the configurations of the vertices, edges, and faces of $P$. [As in Section 2, this is not true if $P$ is not SN, even if it is normal; examples can be easily given.]

An alternative formulation of Theorems 4 and 5 can be obtained if we regard the given partial tiling $\mathcal{P}'$ as a subset of a tiling $\mathcal{P}$, and distinguish the polyhedra $Q$ in $N^*(P, \mathcal{P})$ that share a face with $P$, those that share only an edge with $P$, and those that share only a vertex with $P$. All three types of $Q$'s will be called $\alpha$-adjacent to $P$; those that share a face or edge with $P$ will be called $\beta$-adjacent to $P$; and those that share a face with $P$ will be called $\gamma$-adjacent to $P$. [$\alpha$-, $\beta$-, or $\gamma$-connectedness is the transitive closure of $\alpha$-, $\beta$-, or $\gamma$-adjacency.] We denote the set of polyhedra in $\mathcal{P}'$ that are $\alpha$-adjacent to $P$ by $N^*_\alpha(P)$; the set of polyhedra in $\mathcal{P} - \mathcal{P}'$ that are $\beta$-adjacent to $P$ by $\overline{N}^*_\beta(P)$; and the set of polyhedra in $\mathcal{P} - \mathcal{P}'$ that are $\gamma$-adjacent to $P$ by $\overline{N}^*_\gamma(P)$. In terms of this notation, we can restate Theorems 4 and 5 as follows:

**Theorem 4′** *$P$ is simple if and only if it satisfies the following two conditions:*

1. *$N^*_\alpha(P)$ is nonempty and $\alpha$-connected*

2. *Exactly one $\gamma$-component of $\overline{N}^*_\beta(P)$ intersects $\overline{N}^*_\gamma(P)$.*

[Note that by condition 2, $\overline{N}^*_\gamma(P)$ cannot be empty; hence $P$ must have at least one bare face. Condition 2 then readily implies that the set of bare faces of $P$ is face-connected.]

**Theorem 5'** *When P is deleted,*

1. *The change in the number of components is one less than the number of $\alpha$-components in $N_\alpha^*(P)$.*

2. *The change in the number of tunnels is one less than the number of $\gamma$-components of $\overline{N}_\beta^*(P)$ that intersect $\overline{N}_\gamma^*(P)$ when $\overline{N}_\gamma^*(P)$ is nonempty, and zero otherwise.*

3. *The change in the number of cavities is one when $\overline{N}_\gamma^*(P)$ is empty, and zero otherwise.*

## 3.3 Efficient computation

As we have seen, the numbers of components, tunnels and cavities in $N(P, \mathcal{P})$ are always 1, 0 and 0 respectively. Thus the numbers $\xi(P, \mathcal{P})$, $\eta(P, \mathcal{P})$ and $\delta(P, \mathcal{P})$ of components, tunnels and cavities in $N^*(P, \mathcal{P})$ define local measures of topological change when $P$ is deleted. Let $simple(P, \mathcal{P})$ be a predicate which has the value 1 when $P$ is simple and 0 otherwise. $\xi(P, \mathcal{P})$, $\eta(P, \mathcal{P}), \delta(P, \mathcal{P})$ and $simple(P, \mathcal{P})$ will be referred to as the local topological parameters of $P$. In this section we develop an efficient approach to computing these parameters. For brevity, in this section the configuration of a face, edge, or vertex refers to whether it is shared or bare.

**Proposition 8** *If a face $f$ of $P$ is shared then $\xi(P, \mathcal{P})$ is independent of the configurations of the vertices or edges $s \subset f$.*

**Proof:** Analogous to that of Proposition 3. $\qquad\square$

We can similarly prove

**Proposition 9** *If an edge $e$ of $P$ is shared then $\xi(P, \mathcal{P})$ is independent of the configurations of the vertices of $e$.*

**Proposition 10** *If a face $f$ of $P$ is shared then $\eta(P, \mathcal{P})$ is independent of the configurations of the vertices or edges $s \subset f$.*

**Proof:** According to Corollary 3, $\eta(P, \mathcal{P})$ is independent of the configurations of the vertices of $P$. To establish the proposition we shall show that $\eta(P, \mathcal{P})$ is independent of the configurations of all edges of $f$. Let $e$ be such an edge and let $P_e$ be a polyhedron (if any) such that $P_e \cap P = e$.

According to Theorem 3, $\eta(P, \mathcal{P})$ is zero when $O(P, \mathcal{P})$ is empty, and otherwise it is one less than the number of face components of $O(P, \mathcal{P})$, so that $\eta(P, \mathcal{P})$ is also zero when $O(P, \mathcal{P})$ contains exactly one face. Thus to prove the proposition we need only show that $\eta(P, \mathcal{P})$ is independent of the configuration of $e$ when $O(P, \mathcal{P})$ contains two or more faces. For this purpose we show that two faces of $O(P, \mathcal{P})$ are face-connected in $N^*(P, \mathcal{P} - \{P_e\})$ if and only if they are face-connected in $N^*(P, \mathcal{P})$. Suppose there were two faces $x, y$ in $O(P, \mathcal{P})$ that were face-connected in $N^*(P, \mathcal{P})$ but not in $N^*(P, \mathcal{P} - \{P_e\})$. Then there must exist an alternating sequence of bare faces and bare edges of $P$, say $x = f_0, e_0, f_1, \cdots, e_{n-1}, f_n = y$,

11

such that $interior(f_i)$ and $interior(f_{i+1})$ are adjacent to $interior(e_i)$ for $0 \leq i < n$, and $e_j = e$ for some $j$, $0 \leq j < n$. Now for any edge $z$ of a polyhedron $Q$, $interior(z)$ is adjacent to the interiors of exactly two faces of $Q$. By assumption, face $f$ is shared and $e$ is an edge of $f$, so that $interior(e)$ is adjacent to $interior(f)$. Therefore $P$ cannot have two bare faces whose interiors are adjacent to $interior(e)$, contradiction. $\square$

By Proposition 6, $\delta(P, \mathcal{P})$ depends on the configurations of the faces of $P$, and is independent of the configurations of the edges and vertices of $P$. By Propositions 8 and 10, $\xi(P, \mathcal{P})$ and $\eta(P, \mathcal{P})$ are independent of the configurations of the edges belonging to shared faces of $T$. According to Corollary 3, $\eta(P, \mathcal{P})$ is independent of the configurations of the vertices of $P$. We also see from Propositions 8 and 9 that $\xi(P, \mathcal{P})$ is independent of the configurations of the vertices belonging to shared faces or shared edges. Based on this, we have the following definition and Proposition:

**Definition 3** *An edge $e$ of $P$ is called* trapped *if it belongs to a shared face of $P$; otherwise, it is called* free. *A vertex $v$ of $P$ is called* trapped *if it belongs to a shared face or edge of $P$; otherwise, it is called* free.

**Proposition 11** *$\xi(P, \mathcal{P})$, $\eta(P, \mathcal{P})$ and $\delta(P, \mathcal{P})$ are independent of the configurations of the trapped edges and vertices of $P$.*

Since $simple(P, \mathcal{P})$ depends on $\xi(P, \mathcal{P})$, $\eta(P, \mathcal{P})$ and $\delta(P, \mathcal{P})$ (specifically, $simple(P, \mathcal{P})$ is 1 if and only if $\xi(P, \mathcal{P})$, $\eta(P, \mathcal{P})$ and $\delta(P, \mathcal{P})$ are 1, 0 and 0, respectively), we also have

**Corollary 4** *The predicate $simple(P, \mathcal{P})$ is independent of the configurations of the trapped edges and vertices of $P$.*

**Lemma 8** *Each free, shared vertex of $P$ contributes one component to $N^*(P, \mathcal{P})$.*

**Proof:** Analogous to that of Lemma 3. $\square$

**Lemma 9** *If the edges of $P$ are all trapped then $\xi(P, \mathcal{P}) = 1$.*

**Proof:** Since all edges of $P$ are trapped, $P$ has at least one shared face, so that $\xi(P, \mathcal{P}) \geq 1$. Suppose $\xi(P, \mathcal{P}) > 1$ so that $\mathcal{B}_s(P, \mathcal{P})$ is not connected. Now $\mathcal{B}(P)$ is topologically equivalent to a sphere and $\mathcal{B}_s(P, \mathcal{P}) \subset \mathcal{B}(P)$. Therefore $\mathcal{B}_s(P, \mathcal{P})$ not connected implies that $\mathcal{B}_b(P, \mathcal{P}) = \mathcal{B}(P) - \mathcal{B}_s(P, \mathcal{P})$ contains a tunnel, so that $\mathcal{B}_b(P, \mathcal{P})$ contains a closed curve $c$ that is not reducible to a point in $\mathcal{B}_b(P, \mathcal{P})$. Suppose $c$ passes through the interiors of exactly $n$ faces, edges and/or vertices of $P$ all of which are contained in $\mathcal{B}_b(P, \mathcal{P})$, say $s_1, s_2, \ldots, s_n$ where $n$ is as small as possible. Evidently $n > 1$. [The interior of a face, edge, or vertex is simply connected; hence a simple closed curve entirely contained in the interior of a single face, edge, or vertex is always reducible to a point.] It is not hard to see that at least one of the $s_i$'s must be either an edge or a vertex. [Indeed, the interiors of two faces are always disjoint, so that $c$ cannot pass directly from the interior of one face to the interior of another.] Suppose $s_i$ is a vertex. Let $e$ be any edge that contains $s_i$. If $interior(e)$ meets $\mathcal{B}_s(P, \mathcal{P})$, by normality $e$ must be contained in $\mathcal{B}_s(P, \mathcal{P})$; hence $s_i \subseteq \mathcal{B}_s(P, \mathcal{P})$, contradicting the fact that

12

*interior* $s_i \subseteq \mathcal{B}_b(P, \mathcal{P})$. Hence *interior(e)* must be contained in $\mathcal{B}_b(P, \mathcal{P})$. Thus both the faces containing $e$ are bare so that $e$ cannot be trapped, contradiction. A similar contradiction can be derived if $s_i$ is an edge. Hence $\mathcal{B}_b(P, \mathcal{P})$ cannot contain a closed curve that is not reducible to a point in $\mathcal{B}_b(P, \mathcal{P})$; thus $\mathcal{B}_b(P, \mathcal{P})$ cannot contain a tunnel. Therefore $\mathcal{B}_s(P, \mathcal{P})$ is connected and hence $\xi(P, \mathcal{P}) \leq 1$. $\qquad\square$

To conclude this section, we describe the general structure of an algorithm that computes the local topological parameters $\xi(P, \mathcal{P})$, $\eta(P, \mathcal{P})$, $\delta(P, \mathcal{P})$ and $simple(P, \mathcal{P})$. We treat the following cases, depending on the presence of shared faces in $P$.

**Case 1:** All faces of $P$ are shared.
In this case all the edges and vertices are trapped and no more computation is needed. The local topological parameters have the following values:

$$\xi(P, \mathcal{P}) = 1, \quad \eta(P, \mathcal{P}) = 0, \quad \delta(P, \mathcal{P}) = 1, \quad simple(P, \mathcal{P}) = 0.$$

**Case 2:** Some faces of $P$ are bare but all edges of $\mathcal{P}$ are trapped.
In this case by Lemma 9 $\xi(P, \mathcal{P}) = 1$. Since some faces are bare, by Theorem 5 we must have $\delta(P, \mathcal{P}) = 0$. As regards $\eta(P, \mathcal{P})$, note that since every edge is trapped, two bare faces cannot intersect in an edge; hence no two bare faces are face-adjacent, so the number of face components in $O(P)$ is equal to the number of bare faces. Hence by Theorem 5, $\eta(P, \mathcal{P})$ is one less than the number of bare faces. Therefore, $simple(P, \mathcal{P}) = 1$ iff the number of bare faces is 1, and $simple\ (P, \mathcal{P}) = 0$ otherwise.

**Case 3:** Some or all faces of $P$ are bare and some edges of $P$ are free. [Note that if all the faces are bare, all the edges must be free.]
Here again, $\delta(P, \mathcal{P}) = 0$. $\xi(P, \mathcal{P})$ is the number of components in the union of the shared faces and free shared edges of $P$, plus the number of free shared vertices of $P$ (Theorem 5 and Lemma 8). As for $\eta(P, \mathcal{P})$, by Theorem 3 it is one less than the number of face components in $O(P)$. [Unfortunately, the computation of $\xi(P, \mathcal{P})$ and $\eta(P, \mathcal{P})$ can be complex.] Finally, $simple(P, \mathcal{P}) = 1$ if $\xi(P, \mathcal{P}) = 1$ and $\eta(P, \mathcal{P}) = 0$, and $simple(P, \mathcal{P}) = 0$ otherwise.

### 3.4 An example: Partial tilings by cubes

The computation of the measures of topological change is complicated even if $\mathcal{P}'$ is a partial tiling derived from a regular tessellation. In this section we describe an algorithm for computing these measures in the case of a partial cubic tiling.

In a cubic tessellation $\mathcal{P}$, every cube $P$ has twenty six $\alpha$-neighbors, eighteen $\beta$-neighbors and six $\gamma$-neighbors. An $\alpha$-neighbor that is not a $\beta$-neighbor will be called a *vertex neighbor*; a $\beta$-neighbor that is not a $\gamma$-neighbor will be called an *edge neighbor*; and a $\gamma$-neighbor will be called a *face neighbor*. We denote the cubes in the neighborhood of $P$ as shown in Figure 2.

In this section we give an efficient algorithm that computes the local topological parameters of $P$ in a partial tiling $\mathcal{P}'$ that is a subset of the cubic tessellation $\mathcal{P}$. We first determine the configurations of the six face neighbors of $P$. [In the remainder of this section, the *configuration* of a cube specifies whether the cube is in $\mathcal{P}'$ or in $\mathcal{P} - \mathcal{P}'$. A cube will be referred to as *black* if it belongs to $\mathcal{P}'$; otherwise, it will be referred as *white*. For brevity, $\xi$, $\eta$, $\delta$ and *simple* will refer to $\xi(P, \mathcal{P}')$, $\eta(P, \mathcal{P}')$, $\delta(P, \mathcal{P}')$ and $simple(P, \mathcal{P}')$, respectively.]

13

There are sixty-four possible configurations of the six face neighbors; they can be grouped into the following ten cases depending upon the number and relative positions of the black face neighbors of $P$:

**Case 1:** *All six face neighbors are black.*
Only one face neighbor configuration belongs to this category. All edge and vertex neighbors are trapped. No further computation is necessary and the values of the local topological parameters are $\xi = 1$; $\eta = 0$; $\delta = 1$; *simple* $= 0$.

**Case 2:** *Five face neighbors are black.*
Six face neighbor configurations belong to this category. All edge and vertex neighbors are trapped. No further computation is necessary and the values of the local topological parameters are $\xi = 1$; $\eta = 0$; $\delta = 0$; *simple* $= 1$.

**Case 3:** *Two pairs of opposite face neighbors are black.*
Three face neighbor configurations belong to this category. All edge and vertex neighbors are trapped. No further computation is necessary and the values of the local topological parameters are $\xi = 1$; $\eta = 1$; $\delta = 0$; *simple* $= 0$.

**Case 4:** *One pair of opposite and two non-opposite face neighbors are black.*
Twelve face neighbor configurations belong to this category. All vertex neighbors are trapped, but one edge neighbor is free. The values of the local topological parameters are as follows: $\xi = 1$; $\eta = 0$ if the free edge neighbor is white and $=1$ otherwise; $\delta = 0$; *simple* $= 1$ if the free edge neighbor is white, and $= 0$ otherwise.

**Case 5:** *One pair of opposite face neighbors and one other face neighbor are black.*
Twelve face neighbor configurations belong to this category. All vertex neighbors are trapped, but two edge neighbors are free. The values of the local topological parameters are as follows: $\xi = 1$; $\eta =$ one less than the number of black and free edge neighbors; $\delta = 0$; *simple* $= 1$ if both the free edge neighbors are white, and $= 0$ otherwise.

**Case 6:** *Three non-opposite face neighbors are black.*
Eight face neighbor configurations belong to this category. Three edge neighbors are free, and when all of them are white, the vertex neighbor at their intersection is also free. We determine the configurations of the three free edge neighbors and determine the values of the local topological parameters as follows:

**Case 6.1:** *At least one of the free edge neighbors is black.*
In this case the vertex neighbor at the intersection of the free edge neighbors is trapped and we do not need to know its configuration. The values of the local topological parameters are as follows: $\xi = 1$; $\eta =$ one less than the number of black and free edge neighbors; $\delta = 0$; *simple* $= 1$ if exactly one free edge neighbor is black, and $= 0$ otherwise.

**Case 6.2:** *All three free edge neighbors are white.*
Here the vertex neighbor at the intersection of the free edge neighbors is free and we need to know its configuration. The values of the local topological parameters are as follows: $\xi = 1$ if the free vertex neighbor is white, and $= 2$ otherwise; $\eta = 0$; $\delta = 0$; and *simple* $= 1$ if the free vertex neighbor is white, and $= 0$ otherwise.

**Case 7:** *Two opposite face neighbors are black.*
Three face neighbor configurations belong to this category. Here, all vertex neighbors are trapped, but four edge neighbors are free. Depending on the configurations of the free edge neighbors, the values of the local topological parameters are determined as follows:

14

**Case 7.1:** *At least one free edge neighbor is black.*
The values of the local topological parameters are: $\xi = 1$; $\eta =$ one less than the number of black free edge neighbors; $\delta = 0$; $simple = 1$ if exactly one free edge neighbor is black, and $= 0$ otherwise.

**Case 7.2:** *All four free edge neighbors are white.*
The values of the local topological parameters are: $\xi = 2$; $\eta = 0$; $\delta = 0$; and $simple = 0$.

**Case 8:** *Two non-opposite face neighbors are black.*
Twelve face neighbor configurations belong to this category. Here, five edge neighbors are free and depending upon their configurations, two more vertex neighbors may be free. To describe this case, let us consider an example (see Figure 2). Suppose face neighbors $F_3$ and $F_5$ are black and all other face neighbors are white. Among the free edge neighbors, the blackness of $E_{10}$ leaves both vertex neighbors $V_4$ and $V_8$ [which could be free depending on the configurations of the free edge neighbors] trapped. Therefore, we first determine the configuration of $E_{10}$; the values of the local topological parameters are then determined as follows:

**Case 8.1:** *Edge neighbor $E_{10}$ is black.*
Both vertex neighbors $V_4$ and $V_8$ are trapped, so we do not need to know the configuration of any vertex neighbor. However, the values of the local topological parameters depend on the configurations of the other four free edge neighbors $E_2$, $E_3$, $E_5$ and $E_8$. At this stage, we use a lookup table to determine the values of $\xi$ and $\eta$. The lookup table contains entries for all possible configurations of $E_2$, $E_3$, $E_5$ and $E_8$; for each entry, it contains precalculated values of $\xi$, and $\eta$ (according to Theorem 5') and of *simple* (according to Theorem 4'); note that $\delta$ is always zero.

**Case 8.2:** *Edge neighbor $E_{10}$ is white.*
Here, the configuration of $E_{10}$ is not enough to determine whether the two vertex neighbors $V_4$ and $V_8$ are trapped; this, and the values of the local topological parameters, also depend on the configurations of the other four free edge neighbors $E_2$, $E_3$, $E_5$ and $E_8$; At this stage, we use a lookup table. Again, this table contains entries for all possible configurations of $E_2$, $E_3$, $E_5$ and $E_8$; and for each entry it contains precalculated values of (1) the number of $\alpha$-components $X$ in the set of black face neighbors $F_3$ and $F_5$ and black edge neighbors among $E_2$, $E_3$, $E_5$ and $E_8$; (2) $\eta$; and (3) a two-bit number $Y$ that indicates the free vertex neighbors among $V_4$ and $V_8$. At run time, a two-bit number $Z$ is generated that indicates the configurations of $V_4$ and $V_8$. Finally, $X$ plus the number of "1" bits in $Y \wedge Z$ determines the value of $\xi$; the value of $\delta$ is always 0. Finally, *simple* $= 1$ when $\xi = 1$ and $\eta = 0$; and $= 0$ otherwise.

**Case 9:** *Only one face neighbor is black.*
Six face neighbor configurations belong to this category. Here, eight edge neighbors are free and depending upon their configurations, four more vertex neighbors may be free. To describe this case, let us consider an example (see Figure 2). Suppose face neighbor $F_3$ is black and all other face neighbors are white. Edge neighbors $E_1$, $E_2$, $E_3$, $E_4$, $E_5$, $E_8$, $E_9$ and $E_{10}$ are free. Among them, we can find two sets $\{E_5, E_8\}$ and $\{E_9, E_{10}\}$ such that, independent of the configuration of the other member of the same set, the blackness of one edge neighbor leaves two vertex neighbors [which could be free, depending on the configurations of the free edge neighbors] trapped. We randomly select the first set and determine the configurations

15

of $E_5$ and $E_8$. Depending on how many of them are black, the values of the local topological parameters are determined as follows:

**Case 9.1:** *Both $E_5$ and $E_8$ are black.*
All four vertex neighbors $V_1$, $V_4$, $V_5$ and $V_8$ are trapped and we do not need to know the configuration of any vertex neighbor. However, the values of the local topological parameters depend on the configurations of the other six free edge neighbors $E_1$, $E_2$, $E_3$, $E_4$, $E_9$ and $E_{10}$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.1.

**Case 9.2:** *Exactly one of $E_5$ and $E_8$ is black.*
Suppose $E_5$ is black so that the two vertex neighbors $V_1$ and $V_4$ are trapped. The configurations of $E_5$ and $E_8$ are not enough to determine whether the other two vertex neighbors $V_5$ and $V_8$ are trapped; this, and the values of the local topological parameters, depend on the configurations of the other six free edge neighbors $E_1$, $E_2$, $E_3$, $E_4$, $E_9$ and $E_{10}$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.2.

**Case 9.3:** *Neither $E_5$ nor $E_8$ is black.*
Here, the configurations of $E_5$ and $E_8$ are not enough to determine whether the four vertex neighbors $V_1$, $V_4$, $V_5$ and $V_8$ are trapped; this, and the values of the local topological parameters, depend on the configuration of other six free edge neighbors $E_1$, $E_2$, $E_3$, $E_4$, $E_9$ and $E_{10}$. At this stage, we could use a lookup table; however, it should be noted that we can find two edge neighbors $E_9$, $E_{10}$ such that the blackness of one of them, independent of the configuration of the other, leaves two vertex neighbors trapped which were not yet known to be trapped. Before using a lookup table, we determine the configurations of these edge neighbors. Depending on how many of them are black, the values of the local topological parameters are determined as follows:

**Case 9.3.1:** *Both $E_9$ and $E_{10}$ are black.*
All four vertex neighbors $V_1$, $V_4$, $V_5$ and $V_8$ are trapped and we do not need to know the configuration of any vertex neighbor. However, the values of the local topological parameters depend on the configurations of the remaining four free edge neighbors $E_1$, $E_2$, $E_3$ and $E_4$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.1.

**Case 9.3.2:** *Exactly one of $E_9$ and $E_{10}$ is black.*
Suppose $E_9$ is black so that the two vertex neighbors $V_1$ and $V_5$ are trapped. The configurations of $E_9$ and $E_{10}$ are not enough to determine whether the other two vertex neighbors $V_4$ and $V_8$ are trapped; this, and the values of the local topological parameters, depend on the configurations of the remaining four free edge neighbors $E_1$, $E_2$, $E_3$ and $E_4$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.2.

**Case 9.3.3:** *Neither $E_9$ nor $E_{10}$ is black.*
Here, the configuration of $E_9$ and $E_{10}$ is not enough to determine whether the four vertex neighbors $V_1$, $V_4$, $V_5$ and $V_8$ are trapped; this, and the values of local topological parameters, depend on the configuration of the remaining four free edge neighbors $E_1$, $E_2$, $E_3$ and $E_4$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.2.

**Case 10:** *No face neighbor is black.*

Only one face neighbor configuration belongs to this category. Here, all twelve edge neighbors are free, and depending upon their configurations, all eight vertex neighbors may be free. Obviously this is the most difficult case to handle. ¿From the twelve edge neighbors we can construct three sets $\{E_1, E_2, E_3, E_4\}$, $\{E_5, E_6, E_7, E_8\}$, and $\{E_9, E_{10}, E_{11}, E_{12}\}$, such that, independent of the configurations of the other three members of the same set, the blackness of one edge neighbor in a set leaves two vertex neighbors trapped. We randomly select the first set $\{E_1, E_2, E_3, E_4\}$, and determine their configurations. Depending on which of them are black, the values of the local topological parameters are determined as follows:

**Case 10.1:** *$E_1$, $E_2$, $E_3$ and $E_4$ are all black.*

All eight vertex neighbors are trapped and we do not need to know their configurations. However, the values of the local topological parameters depend on the configurations of the remaining eight free edge neighbors $E_5$, $E_6$, $E_7$, $E_8$, $E_9$, $E_{10}$, $E_{11}$ and $E_{12}$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.1.

**Case 10.2:** *Three of $E_1$, $E_2$, $E_3$ and $E_4$ are black.*

Suppose $E_1$, $E_2$ and $E_3$ are black, so that the six vertex neighbors $V_1$, $V_2$, $V_3$, $V_4$, $V_7$ and $V_8$ are trapped. The configurations of $E_1$, $E_2$, $E_3$ and $E_4$ are not enough to determine whether the other two vertex neighbors $V_5$ and $V_6$ are trapped; this, and the values of local topological parameters, depend on the configurations of the remaining eight free edge neighbors $E_5$, $E_6$, $E_7$, $E_8$, $E_9$, $E_{10}$, $E_{11}$ and $E_{12}$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.2.

**Case 10.3:** *Two of $E_1$, $E_2$, $E_3$ and $E_4$, whose intersections with $P$ are noncoplanar, are black.*

Suppose $E_1$ and $E_3$ are black so that the four vertex neighbors $V_1$, $V_2$, $V_7$ and $V_8$ are trapped. The configurations of $E_1$, $E_2$, $E_3$ and $E_4$ are not enough to determine whether the other four vertex neighbors $V_3$, $V_4$, $V_5$ and $V_6$ are trapped; this, and the values of the local topological parameters, depend on the configuration of the remaining eight free edge neighbors $E_5$, $E_6$, $E_7$, $E_8$, $E_9$, $E_{10}$, $E_{11}$ and $E_{12}$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.2.

**Case 10.4:** *Two of $E_1$, $E_2$, $E_3$ and $E_4$, whose intersections with $P$ are coplanar, are black.*

Suppose $E_1$ and $E_2$ are black, so that the four vertex neighbors $V_1$, $V_2$, $V_3$ and $V_4$ are trapped. The configurations of $E_1$, $E_2$, $E_3$ and $E_4$ are not enough to determine whether the other four vertex neighbors $V_5$, $V_6$, $V_7$ and $V_8$ are trapped; this, and the values of the local topological parameters, depend on the configuration of the remaining eight free edge neighbors $E_5$, $E_6$, $E_7$, $E_8$, $E_9$, $E_{10}$, $E_{11}$ and $E_{12}$. At this stage, we could use a lookup table; however, it should be noted that we can find two edge neighbors $E_7$ and $E_8$ such that the blackness of one of them, independent of the configuration of the other, leaves two vertex neighbors trapped which are not yet known to be trapped. Before using a lookup table, we determine the configurations of these edge neighbors. Depending on how many of them are black, the values of the local topological parameters are determined as follows:

**Case 10.4.1:** *Both $E_7$ and $E_8$ are black.*

17

Here, all eight vertex neighbors are trapped and we do not need to know their configurations. However, the values of the local topological parameters depend on the configurations of the remaining six free edge neighbors $E_5$, $E_6$, $E_9$, $E_{10}$, $E_{11}$ and $E_{12}$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table is exactly the same as in Case 8.1.

**Case 10.4.2:** *Exactly one of $E_7$ and $E_8$ is black.*

Suppose $E_7$ is black, so that the six vertex neighbors $V_1$, $V_2$, $V_3$, $V_4$, $V_6$ and $V_7$ are trapped. The configurations of $E_7$ and $E_8$ are not enough to determine whether the other two vertex neighbors $V_5$ and $V_8$ are trapped; this, and the values of the local topological parameters, depend on the configurations of the remaining six free edge neighbors $E_5$, $E_6$, $E_9$, $E_{10}$, $E_{11}$ and $E_{12}$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.2.

**Case 10.4.3:** *Neither $E_7$ nor $E_8$ is black.*

Here, the configurations of $E_7$ and $E_8$ are not enough to determine whether any of the four vertex neighbors $V_5$, $V_6$, $V_7$ and $V_8$ is trapped; this, and the values of the local topological parameters, depend on the configurations of the remaining six free edge neighbors $E_5$, $E_6$, $E_9$, $E_{10}$, $E_{11}$ and $E_{12}$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.2.

**Case 10.5:** *Only one of $E_1$, $E_2$, $E_3$ and $E_4$ is black.*

Suppose $E_1$ is black, so that the two vertex neighbors $V_1$ and $V_2$ are trapped. The configurations of $E_1$, $E_2$, $E_3$ and $E_4$ are not enough to determine whether the other six vertex neighbors $V_3$, $V_4$, $V_5$, $V_6$, $V_7$ and $V_8$ are trapped; this, and the values of the local topological parameters, depend on the configurations of the remaining eight free edge neighbors $E_5$, $E_6$, $E_7$, $E_8$, $E_9$, $E_{10}$, $E_{11}$ and $E_{12}$. At this stage, we can find two edge neighbors $E_7$ and $E_8$ such that the blackness of one of them, independent of the configuration of the other, makes two vertex neighbors trapped which were not yet known to be trapped. Before using a lookup table, we determine the configurations of these two edge neighbors. Depending on how many of them are black, the values of the local topological parameters are determined as follows:

**Case 10.5.1:** *Both $E_7$ and $E_8$ are black.*

Here, six vertex neighbors $V_1$, $V_2$, $V_5$, $V_6$, $V_7$ and $V_8$, are trapped and we do not need to know their configurations. The configurations of $E_7$ and $E_8$ are not enough to determine whether the other two vertex neighbors $V_3$ and $V_4$ are trapped; this, and the values of the local topological parameters, depend on the configurations of the remaining six free edge neighbors $E_5$, $E_6$, $E_9$, $E_{10}$, $E_{11}$ and $E_{12}$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.2.

**Case 10.5.2:** *Exactly one of $E_7$ and $E_8$ is black.*

Suppose $E_7$ is black, so that the four vertex neighbors $V_1$, $V_2$, $V_6$ and $V_7$ are trapped. The configurations of $E_7$ and $E_8$ are not enough to determine whether the other four vertex neighbors $V_3$, $V_4$, $V_5$ and $V_8$ are trapped; this, and the values of the local topological parameters, depend on the configurations of the remaining six free edge neighbors $E_5$, $E_6$, $E_9$, $E_{10}$, $E_{11}$ and $E_{12}$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.1. Note

that the blackness of $E_{10}$ makes two vertex neighbors trapped which were not yet known to be trapped. This allows us to define another level of cases and improve the efficiency of the algorithm. Unfortunately, it also makes the description more complicated.

**Case 10.5.3:** *Neither $E_7$ nor $E_8$ is black.*
Here, the configurations of $E_7$ and $E_8$ are not enough to determine whether any of the six vertex neighbors $V_3$, $V_4$, $V_5$, $V_6$, $V_7$ and $V_8$ is trapped; this, and the values of the local topological parameters, depend on the configurations of the remaining six free edge neighbors $E_5$, $E_6$, $E_9$, $E_{10}$, $E_{11}$ and $E_{12}$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.2.

**Case 10.6:** *None of $E_1$, $E_2$, $E_3$ and $E_4$ is black.*
Here, the configurations of $E_1$, $E_2$, $E_3$, $E_4$ are not enough to determine whether any of the eight vertex neighbors is trapped; this, and the values of the local topological parameters, depend on the configurations of the remaining eight free edge neighbors $E_5$, $E_6$, $E_7$, $E_8$, $E_9$, $E_{10}$, $E_{11}$ and $E_{12}$. We determine the configurations of $E_5$, $E_6$, $E_7$ and $E_8$. Depending on which of them are black, the values of the local topological parameters are determined as follows:

**Case 10.6.1:** *All of $E_5$, $E_6$, $E_7$ and $E_8$ are black.*
Here, all eight vertex neighbors are trapped and we do not need to know their configurations. The values of the local topological parameters depend on the configurations of the remaining four free edge neighbors $E_9$, $E_{10}$, $E_{11}$ and $E_{12}$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.1.

**Case 10.6.2:** *Three of $E_5$, $E_6$, $E_7$ and $E_8$ are black.*
Suppose $E_5$, $E_6$ and $E_7$ are black, so that the six vertex neighbors $V_1$, $V_2$, $V_3$, $V_4$, $V_6$ and $V_7$ are trapped. The configurations of $E_5$, $E_6$, $E_7$ and $E_8$ are not enough to determine whether the other two vertex neighbors $V_5$ and $V_8$ are trapped; this, and the values of the local topological parameters, depend on the configurations of the remaining four free edge neighbors $E_9$, $E_{10}$, $E_{11}$ and $E_{12}$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.2.

**Case 10.6.3:** *Two of $E_5$, $E_6$, $E_7$ and $E_8$ whose intersections with $P$ are noncoplanar are black.*
Suppose $E_5$ and $E_7$ are black, so that the four vertex neighbors $V_1$, $V_4$, $V_6$ and $V_7$ are trapped. The configurations of $E_5$, $E_6$, $E_7$ and $E_8$ are not enough to determine whether the other four vertex neighbors $V_2$, $V_3$, $V_5$ and $V_8$ are trapped; this, and the values of the local topological parameters, depend on the configurations of the remaining four free edge neighbors $E_9$, $E_{10}$, $E_{11}$ and $E_{12}$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.2.

**Case 10.6.4:** *Two of $E_5$, $E_6$, $E_7$ and $E_8$ whose intersections with $P$ are coplanar are black.*
Suppose $E_5$ and $E_6$ are black, so that the four vertex neighbors $V_1$, $V_2$, $V_3$ and $V_4$ are trapped. The configurations of $E_5$, $E_6$, $E_7$ and $E_8$ are not enough to determine whether the other four vertex neighbors $V_5$, $V_6$, $V_7$ and $V_8$ are trapped; this, and the values of the local topological parameters, depend on the configurations of the remaining four free edge neighbors $E_9$, $E_{10}$,

$E_{11}$ and $E_{12}$ At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.2.

**Case 10.6.5:** *Exactly one of $E_5$, $E_6$, $E_7$ and $E_8$ is black.*

Suppose $E_5$ is black, so that the two vertex neighbors $V_1$ and $V_4$ are trapped. The configurations of $E_5$, $E_6$, $E_7$ and $E_8$ are not enough to determine whether the other six vertex neighbors $V_2$, $V_3$, $V_5$, $V_6$, $V_7$ and $V_8$ are trapped; this, and the values of the local topological parameters, depend on the configurations of the remaining four free edge neighbors $E_9$, $E_{10}$, $E_{11}$ and $E_{12}$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.2.

**Case 10.6.6:** *None of $E_5$, $E_6$, $E_7$ and $E_8$ is black.*

Here, the configurations of $E_5$, $E_6$, $E_7$ and $E_8$ are not enough to determine whether any of the eight vertex neighbors is trapped; this, and the values of the local topological parameters, depend on the configurations of the remaining four free edge neighbors $E_9$, $E_{10}$, $E_{11}$ and $E_{12}$. At this stage, we use a lookup table to determine the values of $\xi$, $\eta$ and *simple*; $\delta$ is always 0. The form and use of the lookup table are exactly the same as in Case 8.2.

# 4  Concluding remarks

We have seen that in two dimensions, the measures of topological change can be computed efficiently even for arbitrary SN partial tilings; but in three dimensions, their computation can be complex. It would be of interest to give algorithms analogous to that in Section 3.4 for partial tilings derived from other tessellations [2,3], which have some advantages for representing three-dimensional digital images.

# References

1. P.K. Saha and A. Rosenfeld, Strongly normal sets of convex polygons or polyhedra, Technical Report CAR-TR-872, Center for Automation Research, University of Maryland, November 1997.

2. C.Y. Hu and R.W. Hall, Efficient computation of topological functions on the face-centered cubic grid, in C. Arcelli, L.P. Cordella, and G. Sanniti di Baja, eds., *Advances in Visual Form Analysis*, World Scientific, Singapore, 1997, pp. 257–266.

3. J. Serra, Cube, cube-octahedron or rhombododecahedron as bases for 3-D shape description, *ibid.*, pp. 502–519.
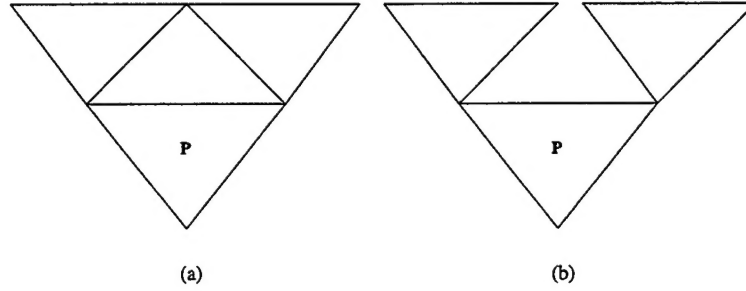
Figure 1: The configurations of the vertices and edges of triangle $P$ are the same in (a) and (b), but $N^*(P, \mathcal{P})$ has only one component in (a), while it has two components in (b). Note that this set of polygons is not SN.
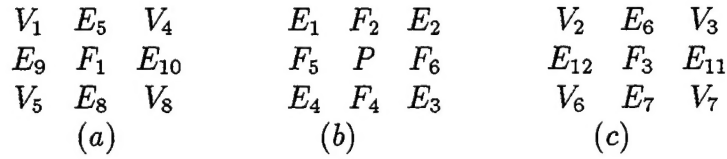
$$
\begin{array}{ccc}
V_1 & E_5 & V_4 \\
E_9 & F_1 & E_{10} \\
V_5 & E_8 & V_8 \\
& (a) &
\end{array}
\qquad
\begin{array}{ccc}
E_1 & F_2 & E_2 \\
F_5 & P & F_6 \\
E_4 & F_4 & E_3 \\
& (b) &
\end{array}
\qquad
\begin{array}{ccc}
V_2 & E_6 & V_3 \\
E_{12} & F_3 & E_{11} \\
V_6 & E_7 & V_7 \\
& (c) &
\end{array}
$$

Figure 2: Notation for the cubes in the neighborhood of $P$. (a) is the layer of cubes above $P$, (b) is the layer containing $P$, and (c) is the layer below $P$. The $F$'s are face neighbors, the $E$'s are edge neighbors, and the $V$'s are vertex neighbors.

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A convex polygon in $R^2$, or a convex polyhedron in $R^3$, will be called a *tile*. A connected set $\mathcal{P}$ of tiles is called a *partial tiling* if the intersection of any two of the tiles is either empty, or is a vertex or edge (in $R^3$: or face) of both. $\mathcal{P}$ is called *strongly normal* (SN) if, for any partial tiling $\mathcal{P}' \subseteq \mathcal{P}$ and any tile $P \in \mathcal{P}'$, the neighborhood $N(P, \mathcal{P})$ of $P$ (the union of the tiles of $\mathcal{P}'$ that intersect $P$) is simply connected. Let $\mathcal{P}$ be SN, and let $N^*(P, \mathcal{P})$ be the excluded neighborhood of $P$ in $\mathcal{P}$ (i.e., the union of the tiles of $\mathcal{P}$, other than $P$ itself, that intersect $P$). We call $P$ *simple* in $\mathcal{P}$ if $N(P, \mathcal{P})$ and $N^*(P, \mathcal{P})$ are topologically equivalent. This paper presents methods of determining, for an SN partial tiling $P$, whether a tile $P \in \mathcal{P}'$ is simple, and if not, of counting the numbers of components and holes (in $R^3$: components, tunnels and cavities) in $N^*(P, \mathcal{P})$.

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